

Endless Games of War: Counting the Deals That Never End

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Abstract

In the card game War with one card of each rank, cards $1, \dots, n$, the higher card wins each turn, and the winning card and then the losing card go under the winner's pile. Some deals never end. Let $a(n)$ be the number of the $n!$ deals of cards $1, \dots, n$ that never end. We compute $a(n)$ exactly for $n \leq 17$, each value with at least two independent programs. Before this, $a(10)$ and the zeros for $n \leq 4$ were known from Spivey, and the zeros at $n = 6, 8, 12$ from searches of all deals by Delahaye and Mathieu. The zeros fall exactly at $n = 1, 2, 3, 4, 6, 8, 12, 16$, the numbers of the form 2^k or $3 \cdot 2^k$. Spivey (2010) proved that every other n has endless games, and his Question 2 asks whether these n have any cycles at all. For each of these n up to 16, an exhaustive search of the positions in which one player holds the highest card on top finds no position of any kind on a cycle. For odd n we prove that every cycle has length $\text{lcm}(n+1, 2a, 2b)$ for some $a, b \geq 1$ with $a + b = (n-1)/2$. Cycles can also be counted without any search, as linear extensions of explicit partial orders. This gives all cycles for odd n , with a closed form when $(n+1)/2$ is prime, and for every n the cycles of the type Spivey constructs. The length of every cycle of that type is divisible by $n + 2^v$, where 2^v is the largest power of 2 dividing n . Spivey's Question 1 asks whether War has cycles of other types than those constructed in the proof of his Theorem 3. It does. The smallest deck with one has 14 cards, and explicit families give such cycles, reached from ordinary deals, for every $n = m \cdot 2^v$ with $m \equiv 3 \pmod{4}$, $m \geq 7$ and $v \geq 1$. Some of them have arbitrarily many categories in the sense of Spivey's original definition. The values $a(n)/n!$ answer Spivey's Question 3 for $n \leq 17$. The sequence is A400411 in the OEIS, and the code and data are public.

1 Introduction

War is a children's card game. The deck is split between two players. On each turn both players turn over their top card, and the higher card takes both. Once the cards are dealt nobody makes a decision, so the deal fixes the whole game, and the question is whether the game has to end. For the full 52-card game, with ties and the "battles" they trigger, this is still open. Delahaye and Mathieu constructed endless games for the 32-card deck with one rule for collecting cards, and by the same method for every even number of ranks from 4 on. Their method does not reach the 52-card deck, with 13 ranks, and their computer searches found no endless 52-card game [3, 2]. For the related game beggar-my-neighbor, whether every game with the standard pack ends was one of Conway's "anti-Hilbert problems", and a game that never ends was reported only in 2024 [7].

Spivey [1] studied the simplest version: one card of each rank, $1, 2, \dots, n$, so there are no ties, and the winner of a turn puts the winning card and then the losing card under their own pile. He showed that some position cycles for every n except possibly $n = 2^k$ and $n = 3 \cdot 2^k$, and that for those other n an ordinary deal can lead to a cycle. His paper ends with seven questions, and the first three are about this game directly:

- Q1. Are there other types of cycles than those constructed in the proof of Theorem 3 of [1]?
- Q2. Do cycles exist for $n = 2^k$ or $n = 3 \cdot 2^k$?
- Q3. What is the probability that an initial deal eventually enters a cycle? (Spivey reports that 395,940 of the $10!$ deals of 10 cards do.)

A cycle is of *Theorem-3 type* if it has the structure of the cycles in the proof of Spivey's Theorem 3. Section 2 gives Spivey's equivalent conditions. Delahaye and Mathieu returned to War in 2025 [2], this time including the single-suit game. They state the conjecture behind Question 2, that cycles exist exactly when n is not of the form 2^k or $3 \cdot 2^k$. Their exhaustive searches for cycles at $n = 2, 3, 4, 6, 8, 12$ found none, and they ran random trials for larger n of that form. Other work on War looks at random rules for collecting the cards [4], the combinatorics of single-suit games [5], and how long a game lasts on average [6]. None of it counts the deals that never end.

Section 2 defines $a(n)$, the number of the $n!$ deals of n cards whose game never ends, and gives it for $n \leq 17$. Every value was computed by at least two programs written independently (Section 3). Our results on Spivey's questions:

- **Q2.** For every $n \leq 16$ of the form 2^k or $3 \cdot 2^k$, no position of n cards lies on a cycle. New here are $n = 16$ and, for $n = 6, 8, 12$, a check of all positions rather than only deals (Section 4).
- **Q1.** There are cycles not of Theorem-3 type. The smallest deck with one has 14 cards. For every $n = m \cdot 2^v$ with $m \equiv 3 \pmod{4}$, $m \geq 7$ and $v \geq 1$ we construct such cycles and show that ordinary deals reach them, and we conjecture that no other n has any (Section 6). They can have arbitrarily many categories in the sense of Section 2: for every $n = 8N + 6$ there is one with $2N + 3$ (Section 6.4). Some of them had already appeared, unidentified, in the code of Delahaye and Mathieu (Section 6.1).
- **Q3.** The exact probability for every $n \leq 17$ (Section 7).

Along the way we prove that for odd n every cycle repeats after exactly $\text{lcm}(n+1, 2a, 2b)$ turns for some $a, b \geq 1$ with $a+b = (n-1)/2$ (Section 5). With the lengths known, the cycles can be counted without any search: a position on a cycle is a labeling of a fixed pattern of places, constrained only by which cards meet, and the number of cycles comes from counts of linear extensions of explicit partial orders. That counts every cycle for odd n (Theorem 5.2), with a closed form when $(n+1)/2$ is prime, and for every n the cycles of Theorem-3 type (Theorem 5.4). Table 2 lists $D(n)$ for odd $n \leq 27$ and $D_3(n)$ for even $n \leq 40$. Khovanova and Pathak use linear extensions in the same way to count the deals that follow a given sequence of wins [5, Sec. 6]. On a cycle that sequence is forced.

Following single cards through a cycle of Theorem-3 type shows that its length is divisible by $n + 2^v$, where 2^v is the largest power of 2 dividing n (Theorem 5.5). The same divisibility holds for the families of cycles not of Theorem-3 type constructed in Section 6. In those of Theorems 6.3 and 6.5, every card that never wins, card 1 in particular, returns to its place after exactly $n + 2^v$ turns (Theorems 6.11 and 6.13). In the family of Theorem 6.6 the two cards of the weakest letter do (Corollary 6.12). Together with the exhaustive search for $n \leq 16$, these results cover every cycle with at most 16 cards, and we conjecture the rule for every cycle (Section 6.5). Section 8 lists related sequences.

2 The game and the sequence

Rules. There are n cards, numbered 1 to n , and two players, each holding a face-down pile. On each turn both players turn over their top card. The higher card wins the turn, and the winning card and then the losing card go under the winner's pile. A player with no cards loses. This is Spivey's rule, which Delahaye and Mathieu call *rangement naturel*. With a single suit there are no ties.

Deals. A deal is an ordering of the n cards, dealt alternately with the first card to player 1, so player 1 starts with $\lceil n/2 \rceil$ cards and player 2 with $\lfloor n/2 \rfloor$. Neither choice affects the count. Any fixed rule for dealing that gives player 1 $\lceil n/2 \rceil$ cards produces each pair of starting piles of these sizes exactly once as the deck runs over its $n!$ orderings, so every such rule counts the same set of starting positions. And for odd n it does not matter which player receives the extra card, because exchanging the players turns an endless game into an endless game.

Definition 2.1. $a(n)$ is the number of the $n!$ deals of cards $1, \dots, n$ for which the game never ends.

A *position* is the ordered pile of each player. A game either ends or eventually repeats a position, and from then on it repeats forever. We call the repeating part a *cycle* and the number of turns in it the *length* of the cycle. Exchanging the players maps a cycle to a cycle, its *exchange*. A card's *place* is its pile and its depth in it. In a cycle, the *period* of a card is the least p such that the card's place after t turns and after $t + p$ turns agree for every t . The position repeats exactly when every card is back in its place, so the length of a cycle is the least common multiple of the periods of its cards.

n	$a(n)$	$a(n)/n!$	longest game that ends
1	0	0	0
2	0	0	1
3	0	0	3
4	0	0	6
5	30	0.2500	8
6	0	0	15
7	2,304	0.4571	15
8	0	0	26
9	218,680	0.6026	28
10	395,940	0.1091	49
11	28,223,770	0.7071	41
12	0	0	74
13	4,880,546,113	0.7838	57
14	3,790,165,176	0.0435	461
15	1,095,264,882,758	0.8376	80
16	0	0	not computed
17	312,707,435,743,775	0.8792	100

Table 1: The number of deals that never end, their share of all $n!$ deals, and the length in turns of the longest game that does end. Every value was computed by at least two independent programs (Section 3). The longest ending game for $n = 16$ needs all $16!$ deals and was not computed.

The sequence is A400411 in the OEIS [8]. Its zeros are exactly at $n = 1, 2, 3, 4, 6, 8, 12, 16$, the numbers of the form 2^k or $3 \cdot 2^k$ (OEIS A029744 [8]). Of these values, $a(10)$ was published before

[1], as were the zeros for $n \leq 4$ (Spivey shows that a deck of four or fewer cards cannot cycle [1]) and the zeros for $n = 6, 8, 12$ [2].

The rest of the paper studies the cycles themselves, using two notions from Spivey's paper.

Categories. Spivey sorts the cards of a cycle into categories [1, p. 749]. The A cards never lose. The B cards lose only to A cards, the C cards only to A or B cards, and so on: a card that loses sits one category below the lowest category among the cards that beat it. So every card is in a lower category than every card that beats it. In his Section 3 Spivey also uses a numerical labeling [1, p. 752], in which a card is one category below the highest-ranked card that beats it. The two labelings do not always agree (Section 6.1), and we use the first one throughout. Replace each card by the letter of its category, and the game can be played on letters. Spivey's Lemma 1 says that there is an n -card cycle if and only if there is an n -card cycle of letters in which two equal letters never meet.

Spivey's proof of the direction from letters to cards gives more, and we use it often. Suppose a letter game returns to its start and two equal letters never meet on the way. Number the cards so that in every meeting of the letter cycle the card of the stronger letter is the higher card. We call this a numbering that *respects every meeting*. Numbering the cards so that every letter beats every later letter is one such numbering. The numbered game then follows the letter game move for move. Call one trip around the letter cycle a *pass*. Every pass moves the cards among the places by the same permutation, which depends only on the letter game, so after finitely many passes the numbered position itself comes back. The starting position therefore lies on a cycle, and its length is the same for every numbering that respects every meeting.

More generally, a cycle *follows* a letter cycle if its cards can be given letters so that the card with the stronger letter wins every turn and, with each card replaced by its letter, the cycle passes through a position of that letter cycle or of its exchange. The numbered cycles above follow their letter cycle, and every cycle follows the letter cycle of its categories.

Cycles of Theorem-3 type. In the proof of Theorem 3 of [1], Spivey builds cycles for every $n = m \cdot 2^v$ with m odd and $m \geq 5$. His Section 3 shows that the following conditions on a cycle are equivalent [1, p. 752]: (1) the players take turns winning r turns in a row, for some fixed $r \geq 1$; (2) the same with r a power of 2; (3) whenever a card loses, the winner is in the category directly above the loser's; (4) whenever a card loses, the winner is one or two categories above the loser's; (5) the positions have an explicit form, the one found in the cycles of Spivey's Theorem 3. A cycle is of *Theorem-3 type* if it satisfies these conditions. For odd n , Spivey's Theorem 6 shows that every cycle alternates single wins, so every odd- n cycle is of Theorem-3 type.

3 Computation and verification

To compute $a(n)$ we play each of the $n!$ deals until it ends or repeats a position.

Canonical positions. To decide whether *any* position of n cards cycles, we do not need to play every position. A much smaller set is enough.

Lemma 3.1. *Up to exchanging the two players, every cycle of War with n cards passes through a position in which player 1's top card is n .*

Proof. Card n never loses, so it never changes hands. In a cycle neither pile ever empties, and each turn removes the top card of each pile, so every card in a pile reaches the top within as many turns as the pile has cards. In particular card n reaches the top of its holder's pile. Exchanging the players maps cycles to cycles. $\square$

A position with card n on top of player 1's pile is fixed by the order of the other $n - 1$ cards and by how many of them player 1 holds, from 0 to $n - 2$. So there are $(n - 1)!(n - 1)$ such *canonical positions*. For $n = 16$ that is 19,615,115,520,000, against $16! \cdot 15 \approx 3.1 \times 10^{14}$ positions with both piles nonempty, and a scan of the canonical positions finds every cycle. We compared it with scans of *all* positions for every $n \leq 9$ (program `canon_check.py`), and they find the same cycles.

Programs. Every term was computed by at least two programs, written independently from the rules alone and sharing no code:

1. `war.cpp`, a C++ program with OpenMP (Brent's cycle detection, cards packed four bits each into one 64-bit word per pile, player 1 dealt every other card), $2 \leq n \leq 15$, with a variant, `war_all.cpp`, that scans all positions (run for $n = 9$ and 12);
2. `war_floyd.c`, a C program with OpenMP (Floyd's cycle detection, circular byte buffers, player 1 dealt the first $\lceil n/2 \rceil$ cards of each ordering, or the first s cards to scan every position), deals for $2 \leq n \leq 14$ and all positions for $n = 9$ and 12;
3. `war_gpu.py`, a GPU program (Brent's cycle detection, top card in the low four bits of the packed word), deals for $n \leq 15$ and canonical positions for $n \leq 14$ and $n = 16$;
4. `war_gpu_floyd.py`, a second GPU program (Floyd's cycle detection, top card in the high four bits), deals for $2 \leq n \leq 15$ and canonical positions for $2 \leq n \leq 14$ and $n = 16$.

Both CPU programs split the work by the first two cards of an ordering and step through the rest in lexicographic order. Both GPU programs give each thread a run of consecutive orderings, also in lexicographic order. In the canonical scans the GPU programs also test every cycle they reach for Theorem-3 type, using the lengths of its runs of wins (Section 6.1). Plain Python simulations reproduce the terms for $n \leq 10$, and for $n = 1$ they give the second count, beside `war_gpu.py`. All four programs agree on the terms for $2 \leq n \leq 14$, programs 1, 3 and 4 agree on $a(15)$, and programs 3 and 4 agree on the canonical scan for $n = 16$. The programs and their output are in the repository named at the end of this section.

Counting checks. The GPU programs count the deals (or canonical positions) they play and stop with an error unless the total is exactly $n!$ (or $(n - 1)!(n - 1)$). The CPU programs print the total next to the result, and it is exact in every run reported here. For $n = 14$ the two GPU programs play one deal from each pair related by exchanging the players, $n!/2$ deals checked against $n!/2$, and double the count. `war_gpu.py` plays the deals that give card n to player 1. `war_gpu_floyd.py` puts card n at each of player 1's $n/2$ places in turn and orders the other $n - 1$ cards in every way. The CPU programs and `war24.py` (below) play all $n!$ deals.

Agreement with published values. Spivey found $a(10) = 395,940$ by enumeration [1, Question 3]. The code of Delahaye and Mathieu [2] reports 197,970 endless deals among the $10!/2$ deals it enumerates (up to exchanging the players), no endless deal for $n = 12$, and the longest ending games for $n = 2, \dots, 10$ and 12. All of these agree with our values.

Odd n beyond 16. For $17 \leq n \leq 20$ we use `war24.py`, a GPU program that stores five-bit cards in two 64-bit words per pile. Its deal ranks are 64-bit integers, which limits it to 20 cards. It detects cycles with Floyd’s method or, for odd n , with a test by exact repetition that Theorem 5.1 makes possible. At a moment when the next winner holds an even number W of cards, the program records the position. If the players then alternate winning for $\text{lcm}(n+1, W, n-W-1)$ turns, it compares the position with the recorded one: equality proves that the game cycles, and otherwise it records a new position. With this test the program reproduces every term for odd $5 \leq n \leq 13$. With Floyd’s method it reproduces $a(10)$, $a(12)$ and $a(14)$ from all $n!$ deals. The two methods agree on a block of 10^9 consecutive deals of 17 cards (lexicographic ranks 10^{14} to $10^{14} + 10^9 - 1$).

With the test by exact repetition, `war24.py` played all $17! = 355,687,428,096,000$ deals of 17 cards and found $a(17) = 312,707,435,743,775$, with no game left undecided. The longest game that ends has 100 turns. A second program, `war17.py`, written independently of `war24.py`, gives the same two numbers. It keeps card n out of the packed piles and tracks only its depth in its holder’s pile, so that 17 cards fit in four-bit fields, and it detects cycles with Brent’s method on the full position, with no limit on the number of turns. It too played all $17!$ deals.

Runtimes. On one laptop GPU (NVIDIA RTX 5080, 16 GB), $a(14)$ took 2.7 and 4.4 minutes in the two GPU programs, $a(15)$ took 71 and 93 minutes, and the canonical scan for $n = 16$ took 3.8 and 2.1 hours. For $a(17)$, `war17.py` played about 6.14×10^8 deals per second and took 161 hours, and `war24.py` played about 5.26×10^8 deals per second, which is about 188 hours for all $17!$ deals. Code and data: <https://github.com/OrygnsCode/endless-games-of-war>.

4 Question 2: no cycles for $n = 2^k$ and $3 \cdot 2^k$ up to 16

Theorem 4.1 (computer-assisted). *For $n = 1, 2, 3, 4, 6, 8, 12, 16$ no position of War with n cards cycles. In particular $a(n) = 0$ for these n .*

Spivey shows this for $n \leq 4$ in the proof of his Theorem 3 [1, p. 750]. For $n = 6, 8, 12, 16$ both GPU programs scanned every canonical position (Lemma 3.1) and found no cycle. At $n = 16$ that is all 19,615,115,520,000 canonical positions, and both programs confirm the exact count. For $n = 12$, `war_all.cpp` and `war_floyd.c` also scanned all $12! \cdot 11 = 5,269,017,600$ positions, with the same result.

The searches of Delahaye and Mathieu [2] enumerate deals. These scans cover every position, as Question 2 asks, and they add $n = 16$.

In the other direction, Spivey’s Theorem 3 gives a cycle for every n that is not of the form 2^k or $3 \cdot 2^k$, and his construction also gives a deal that enters it. For odd n the cycling position already gives one player a single card more than the other, and for even n Spivey builds a deal with $n/2$ cards each that reaches the cycle [1, p. 751]. So $a(n) > 0$ whenever n is not of the form 2^k or $3 \cdot 2^k$ (the numbers of OEIS A029744), and Theorem 4.1 gives $a(n) = 0$ for the eight such $n \leq 16$. The conjecture of Delahaye and Mathieu recalled in the introduction would make $a(n) = 0$ for all of them.

5 Cycle lengths and counts

5.1 Lengths for odd n

Theorem 5.1. *Let n be odd, and take a cycle of War with n cards at a moment when player 1 is about to win. Then player 1 holds $2a$ cards and player 2 holds $2b + 1$ cards for some $a, b \geq 1$ with $a + b = (n - 1)/2$, and the cycle has length exactly $\text{lcm}(n + 1, 2a, 2b)$.*

Proof. By Spivey's Theorem 6 [1], for odd n a cycle has only two categories (Section 2): A , the cards that never lose, and B , the cards that lose only to A cards. Every turn is won by an A card, and the players alternate winning. So each player plays an A card in the turns it wins and a B card in the turns it loses. Take a moment when player 1 is about to win. From then on player 1's cards come up in turns it wins, loses, wins, ..., so its pile reads $A, B, A, B, \dots$ from the top, and player 2's reads $B, A, B, \dots$. The bottom card of each pile went under it as the loser in that player's last win, so it is a B card. Hence player 1 holds an even number $2a$ of cards and player 2 an odd number $2b + 1$. Player 1 holds an A card, so $a \geq 1$. Player 2 wins the next turn with its second card, which is an A card and not its bottom card, so $b \geq 1$.

Now follow single cards. An A card of player 1 goes under player 1's pile at depth $2a$, and $2a$ turns later it is on top again and wins again, so its place repeats with period exactly $2a$. In the same way an A card of player 2 has period exactly $2b$. A B card goes under player 1's pile at depth $2a + 1$, and $2a + 1$ turns later it is played and loses. It then goes under player 2's pile at depth $2b + 1$, loses again $2b + 1$ turns after that, and is back under player 1's pile. Its period is exactly $2a + 2b + 2 = n + 1$. By Section 2 the length of the cycle is the least common multiple of the periods of its cards, and all three kinds of card occur. $\square$

In particular every cycle length for odd n is a multiple of $n + 1$. The theorem allows only the lengths $\text{lcm}(n + 1, 2a, 2b)$ with $a, b \geq 1$ and $a + b = (n - 1)/2$, and each of them occurs, because giving the A cards the $(n - 1)/2$ highest values always yields a cycle (Section 5.2). For $n = 13$ the pairs $(a, b) = (1, 5), (2, 4), (3, 3)$ give lengths 70, 56, 42, and for $n = 15$ the pairs $(1, 6), (2, 5), (3, 4)$ give 48, 80, 48. The exhaustive runs find these lengths and no others. At $n = 15$ there are 723,147,809,952 endless deals that fall into cycles of length 48 and 372,117,072,806 that fall into cycles of length 80. For $n = 5, 7, 9, 11$ the observed lengths are $\{6\}$, $\{8\}$, $\{20, 30\}$ and $\{12, 24\}$, as predicted, and for $n = 5, 7, 9$ these are also the lengths recorded in the code of Delahaye and Mathieu [2]. Random deals for $n = 17$ and 19 show the predicted sets $\{36, 72, 90, 126\}$ and $\{40, 60, 80, 140\}$ and no other lengths.

5.2 Counting the cycles for odd n

The shape of a cycle is rigid enough to count the cycles directly. Here we count all of them for odd n , and Section 5.3 counts those of Theorem-3 type for every n .

Write $n = 2h + 1$ and fix $a, b \geq 1$ with $a + b = h$. Take a cycle at a moment when player 1 is about to win and player 1 holds $2a$ cards, and number the turns from there, so that player 1 wins the even turns $2s$ and player 2 the odd turns $2s + 1$. Name the cards by when they are played: player 1 plays $A_{s \bmod a}$ at turn $2s$, player 2 plays $A'_{s \bmod b}$ at turn $2s + 1$, and $B_{s \bmod (h+1)}$ is the B card that player 2 plays at turn $2s$. The proof of Theorem 5.1 shows that these names are consistent. An A card of player 1 is back on top every $2a$ turns and one of player 2 every $2b$ turns, and a B card

played by player 2 at turn $2s$ is played by player 1 at turn $2s + 2a + 1$ and by player 2 again at turn $2s + n + 1$. So at turn 0 the piles are

$$\text{player 1: } A_0 B_{b+1} A_1 B_{b+2} \cdots A_{a-1} B_h, \quad \text{player 2: } B_0 A'_0 B_1 A'_1 \cdots A'_{b-1} B_b.$$

Card A_i meets card B_j when some s has $s \equiv i \pmod{a}$ and $s \equiv j \pmod{h+1}$, that is, when $i \equiv j$ modulo $\gcd(a, h+1)$. In the same way A'_i meets B_j (at turn $2s+1$, where player 1 plays $B_{(s-a) \bmod (h+1)}$) when $i \equiv j+a$ modulo $\gcd(b, h+1)$. Let $P_{a,b}$ be the partial order on these n names in which each A or A' is above every B it meets, and let $e(a, b)$ be the number of its linear extensions: the number of ways to give the names the values $1, \dots, n$ so that each A or A' gets a higher value than every B below it. Khovanova and Pathak [5, Sec. 6] count the deals that follow a given sequence of wins by the linear extensions of a similar order on the cards.

Theorem 5.2. *Let $n = 2h + 1 \geq 5$. The number of cycles of War with n cards, counting a cycle and its exchange as one, is*

$$D(n) = \sum_{a=1}^{h-1} \frac{e(a, h-a)}{\text{lcm}(n+1, 2a, 2h-2a)},$$

and the number of these cycles of length ℓ is the sum of the same terms over the a with $\text{lcm}(n+1, 2a, 2h-2a) = \ell$.

Proof. Give the names of the display above the values $1, \dots, n$ and consider the resulting position. Suppose first that the values form a linear extension of $P_{a,b}$, and compare the numbered game with the play in which player 1 wins every even turn and player 2 every odd turn. In that play the names move as described, so at every turn the two cards played are an A or A' and a B that it meets, and the A or A' is the higher. The numbered game therefore follows that play turn by turn, forever, and returns to its start: the position lies on a cycle. Conversely, by Theorem 5.1 and Spivey's Theorem 6, every position on a cycle at a moment when player 1 is about to win arises in this way from some a . Its values form a linear extension, since the cards that meet are exactly the pairs above. So the positions on cycles at which player 1 is about to win with $2a$ cards correspond one to one to the linear extensions of $P_{a, h-a}$.

By Theorem 5.1, a cycle whose player 1 holds $2a$ cards when about to win has length $L = \text{lcm}(n+1, 2a, 2h-2a)$. The players alternate winning, so exactly $L/2$ of its positions have player 1 about to win, and there are $2e(a, h-a)/L$ such cycles. Exchanging the players maps them onto the cycles with a replaced by $h-a$. It never maps a cycle to itself, because card n never changes hands. Summing over a and halving gives $D(n)$, and the same count restricted to one value of L gives the second statement. $\square$

When $h+1 = (n+1)/2$ is prime, the gcds above are all 1 (note $\gcd(h-a, h+1) = \gcd(a+1, h+1)$). Then every A is above every B , and $e(a, h-a) = h!(h+1)!$ for every a , which gives a closed form.

Corollary 5.3. *If $p = (n+1)/2$ is prime, then $D(n) = (p-1)! p! \sum_{a=1}^{p-2} \frac{1}{\text{lcm}(2p, 2a, 2p-2-2a)}$.*

For example, $n = 13$ gives $p = 7$ and $D(13) = 6! 7! (\frac{1}{70} + \frac{1}{56} + \frac{1}{42} + \frac{1}{56} + \frac{1}{70}) = 319,680$. Delahaye and Mathieu [2] observed one instance of the prime case. For $n = 9$ and the pattern $ABAB$ against $BABAB$, all $5! \cdot 4! = 2,880$ ways of giving the A cards the values 6 to 9 and the B cards 1 to

5 produce a cycle. In general $P_{a,b}$ splits into classes of interchangeable names: the B cards by their residues modulo $\gcd(a, h+1)$ and $\gcd(b, h+1)$, and the A and A' cards by theirs. A short dynamic program over the number of names placed from each class (program `odd_formula.py`) computes $e(a, b)$ in a few seconds for every $n \leq 33$. The same argument counts the canonical positions of Lemma 3.1 that lie on a cycle. They are the positions above with $A_0 = n$, so there are $\sum_a e'(a, h-a)$ of them, where e' counts the linear extensions of $P_{a,b}$ without A_0 .

Theorem 5.2 gives $D(n) = 2, 52, 336, 55,592, 319,680, 48,304,896$ for $n = 5, 7, \dots, 15$. For $n \leq 13$ these agree with the exhaustive scans of the canonical positions, including the split by length (144 cycles of length 20 and 192 of length 30 for $n = 9$). For $n = 15$ they agree with the search of Section 8 over the positions allowed by Spivey's Theorem 6, which matches for each (a, b) and so, with Theorem 5.1, gives 42,063,360 cycles of length 48 and 6,241,536 of length 80. The formula and the searches also agree on the numbers 6, 144, 2,160, 169,772, 3,024,000, 369,699,840 of canonical positions on a cycle. For $n = 5, 7, 9$ we also played the positions of all $n!$ labelings of every $P_{a,b}$ (program `slot_check.py`). Each linear extension returns to its start after exactly $\text{lcm}(n+1, 2a, 2b)$ turns, and no other labeling lies on a cycle at all. For $n = 17$ the count $e'(a, h-a)$ above gives the number of canonical positions on cycles for each a . Exhaustive searches of two of these cases by `odd_cycles.c` (Section 8) find the same numbers of positions that return to their start after $\text{lcm}(n+1, 2a, 2b)$ turns: $e'(4, 4) = 1,828,915,200$ and $e'(5, 3) = 3,249,590,400$ (cycle lengths 72 and 90 by Theorem 5.1).

5.3 Cycles of Theorem-3 type

Write $n = m \cdot 2^v$ with m odd, and $M = 2^v$. By Spivey's Theorems 11 and 12, in a cycle of Theorem-3 type the players alternate winning blocks of M turns. (The block length is a power of 2, and n is congruent to it modulo twice it, so it is M .) At a moment when player 1 is about to win a block, player 1 holds a groups of $2M$ cards and player 2 holds M cards followed by b groups of $2M$ cards, each group beginning with an A card, where $2a + 2b + 1 = m$ [1, proof of Theorem 11], and $a, b \geq 1$ by Spivey's Lemma 2. In particular $m \geq 5$, so no cycle of $n = 2^k$ or $3 \cdot 2^k$ cards, if one exists, is of Theorem-3 type. Again a position is a labeling of n fixed places. The *forced play*, in which player 1 wins M turns, then player 2 wins M turns, and so on, moves the contents of the places in a way that does not depend on the values. Let $L_{a,b}$ be the number of turns after which it first restores the starting arrangement, and let $Q_{a,b}$ be the relation in which place x is above place y when, during the first $L_{a,b}$ turns of the forced play, the card that started in x beats the card that started in y . Number the places of a group $0, 1, \dots, 2M-1$ from its A card, so that player 2's first M cards sit in places $M, \dots, 2M-1$. In the forced play a card in place $i < M$ beats the card in place $M+i$ of the other player, whose binary expansion has one more 1. A card moves from place i to place $2i$ (if $i < M$) or $2(i-M)+1$ (if $i \geq M$), which keeps the number of 1s. So each card keeps the number of 1s of its first place, and every beat goes from a place with j ones to one with $j+1$. These levels are Spivey's categories [1, Theorem 13]. In particular $Q_{a,b}$ generates a partial order. Let $e_M(a, b)$ be the number of its linear extensions. For $M = 1$ we recover $\text{lcm}(n+1, 2a, 2b)$, $P_{a,b}$ and $e(a, b)$.

Theorem 5.4. *Let $n = m \cdot 2^v$ with $m \geq 5$ odd, $M = 2^v$ and $h = (m-1)/2$. The number of cycles of Theorem-3 type with n cards, counting a cycle and its exchange as one, is*

$$D_3(n) = \sum_{a=1}^{h-1} \frac{M e_M(a, h-a)}{L_{a, h-a}},$$

and the cycles counted by the term for a have length $L_{a,h-a}$. For odd n , $D_3(n) = D(n)$.

Proof. The argument of Theorem 5.2 applies with blocks in place of turns. If the values form a linear extension of $Q_{a,b}$, the numbered game follows the forced play forever, and the position lies on a cycle whose winners alternate in blocks of M , a cycle of Theorem-3 type. Conversely, the positions of such a cycle at the start of a block of player 1 have the shape above, and the numbered game is the forced play. A position determines all later winners, and the winners of the forced play form the sequence $1^M 2^M 1^M 2^M \dots$, whose least period is $2M$. So if a position of the cycle recurs after q turns, q is a multiple of $2M$. The cycle then contains exactly $L_{a,b}/2M$ positions at the start of a block of player 1, and there are $2Me_M(a,b)/L_{a,b}$ cycles of shape a . Exchanging the players maps shape a to shape $h-a$ and fixes no cycle. For odd n every cycle is of Theorem-3 type by Spivey's Theorem 6. $\square$

The lengths $L_{a,b}$ follow from how single cards travel. As above (and in the proof of Spivey's Theorem 13), a card in place $i < M$ wins its next turn and moves to place $2i$ of a new group of the same player, and a card in place $i \geq M$ loses and moves to place $2(i-M) + 1$ of a new group of the other player. A group made during a block of player 1 is played $2a$ blocks later (its first half) and $2a+1$ blocks later (its second half), since player 1 holds a groups at the start of each of its blocks. For player 2 the same holds with b . So a card in place i of a group of player 1 waits $2a$ blocks if $i < M$ and $2a+1$ blocks if $i \geq M$, and similarly with b for player 2. Following a card until it is back in the same place with the same player gives its *route*. The length of a route in blocks is the sum of these waits, and $L_{a,b}$ is M times the least common multiple of the route lengths over all places and both players.

Theorem 5.5. *Every cycle of Theorem-3 type with n cards has length divisible by $n + 2^v$, where 2^v is the largest power of 2 dividing n . The cards that never win, card 1 among them, are the cards in place $2M-1$ of a group, and each has period exactly $n + 2^v$.*

Proof. From a place i with $M \leq i < 2M-1$ a card moves to $2(i-M) + 1$, whose distance $2(2M-1-i)$ from $2M-1$ is twice the old one. So it reaches a place below M and then wins, and the cards that never win, card 1 among them, are exactly the cards in place $2M-1$ of a group. Such a card loses whenever it is played and goes to place $2M-1$ of a new group of the other player. By the waits above it is played $2a+1$ blocks after reaching player 1 and $2b+1$ blocks after reaching player 2. After $(2a+1) + (2b+1) = m+1$ blocks, that is $M(m+1) = n + 2^v$ turns, it is again in place $2M-1$ of a group of the same player, with the piles in the same phase ($m+1$ is even). In the forced play, where a card goes next depends only on its place and the phase, so its places repeat after $n + 2^v$ turns, and its period divides $n + 2^v$. In the $n + 2^v$ turns after it reaches player 1 it is with player 1 for one unbroken stretch and then with player 2 for one. A period $p < n + 2^v$ would repeat this pattern $(n + 2^v)/p \geq 2$ times within those turns and give at least two stretches with each player. So its period is exactly $n + 2^v$. The length of the cycle is a multiple of the period of each of its cards (Section 2), hence of $n + 2^v$. $\square$

For odd n the routes are $2a$ and $2b$ for the A cards and $n+1$ for the B cards, which is Theorem 5.1 again. For $M=2$ the routes are $2a$, $2b$, $m+1$ and $2m$ blocks, so $L_{a,b} = 2\text{lcm}(2a, 2b, m+1, 2m)$. We checked the route formula against the forced play for every $n = m \cdot 2^v \leq 64$ with $v \leq 3$ (program `t3_period.py`).

Theorem 5.4 gives $D_3(10) = 48$, all of length 60, and $D_3(14) = 25,920$, all of length 112. These agree with the exhaustive counts of Section 8: 48 cycles of 10 cards, all of length 60, and, among

the 26,640 cycles of 14 cards, 25,920 of length 112. The other 720 are not of Theorem-3 type (Section 6.1). So for every $n \leq 16$ the number of cycles is $D_3(n)$, plus 720 at $n = 14$. The lengths also agree with random sampling (Section 6.5). Every cycle of Theorem-3 type found among 10^7 random positions for each of $n = 18, 20, 22, 26, 28, 34, 36, 42, 44, 50$ has one of the lengths $L_{a,b}$, and for seven of these ten n every $L_{a,b}$ occurs. Table 2 lists $D(n)$ for odd n and $D_3(n)$ for even n .

n	$D(n)$	n	$D_3(n)$
5	2	10	48
7	52	14	25,920
9	336	18	66,044,160
11	55,592	20	14,515,200
13	319,680	22	78,382,080,000
15	48,304,896	26	153,127,231,488,000
17	3,588,302,592	28	2,556,649,090,830,336
19	653,644,195,200	30	29,092,074,518,707,200,000
21	6,150,380,544,000	34	32,920,915,208,401,059,840,000
23	15,224,690,054,209,536	36	1,861,237,826,094,366,720,000
25	123,652,308,713,472,000	38	1,033,772,618,064,073,221,734,400,000
27	243,832,544,506,375,372,800	40	229,596,050,715,416,985,600,000

Table 2: Left: the number $D(n)$ of cycles of War with n cards for odd n , from Theorem 5.2. Right: the number $D_3(n)$ of cycles of Theorem-3 type for even n with $m \geq 5$, from Theorem 5.4. It is 0 for the other even n . Both count a cycle and its exchange as one. The values for $n \leq 13$ and $D_3(10)$, $D_3(14)$ were also found by an exhaustive scan of the canonical positions, and $D(15)$ by a search over the positions allowed by Spivey’s Theorem 6.

These theorems count cycles, not deals. The sequence $a(n)$ also counts the deals that reach a cycle after a transient, and we know no way to count those without playing them. Cycles that are not of Theorem-3 type are the subject of Section 6, and Section 6.5 extends Theorem 5.5 to the ones constructed there.

6 Question 1: cycles not of Theorem-3 type

Spivey’s Question 1 asks whether every cycle is of Theorem-3 type (Section 2). It is not. The smallest counterexample has 14 cards, and there are infinite families of them.

6.1 The smallest example has 14 cards

Condition (1) of Section 2 is easy to test: walk once around the cycle and record the lengths of the runs of consecutive wins. By Spivey’s Theorems 11 and 12 the common run length in condition (1) can only be 2^v , so the test checks that every run has length exactly 2^v . The canonical-position scans of Section 3 apply this test to every cycle they find. As a control, the same test given a wrong run length rejects every cycle.

Theorem 6.1 (computer-assisted). *Every War cycle with $n \leq 13$ cards is of Theorem-3 type. For $n = 14$ there are cycles that are not.*

The second statement is the case $k = 1$ of Theorem 6.2 below. For odd n the first statement is Spivey’s Theorem 6, and among even $n \leq 13$ only $n = 10$ has cycles at all (for $n = 6, 8, 12$ there

are none, by Theorem 4.1). So the computation that matters is at $n = 10$: all 287,133 canonical positions for $n = 10$ whose game never ends reach cycles that pass the test. Two GPU programs computed this, and a separate Python program, `cycle_type.py`, checked all 48 cycles. For $n = 14$, 43,669,233 of the 2,872,129,173 canonical positions whose game never ends reach a cycle that fails it. These cycles all have length 96, and the others have length 112. Two independent programs agree on both counts. Counted over deals, 61,184,620 of the $a(14) = 3,790,165,176$ endless deals fall into one of these cycles.

Such cycles have in fact appeared before without being identified. For 22 cards, the code of Delahaye and Mathieu notes a few cycles of more than 1,000 turns among 10^6 random deals [2], but every cycle of Theorem-3 type with 22 cards has length 264 or 528 (the formula for $M = 2$ after Theorem 5.5). The sampling of Section 6.5 finds such cycles at $n = 22$, the longest of length 1,440.

One position on such a cycle, top card first, is

player 1: 1, 13, 11, 9, 4 player 2: 3, 5, 2, 14, 12, 10, 8, 7, 6.

It returns to itself after exactly 96 turns. From this position the winners are, turn by turn, 2, 1, 1, 2, 2, 1, 2, 1 repeated 12 times, so the runs of wins have lengths 1, 2, 2, 1, 1, 1 and condition (1) fails (Figure 1). With the definition of Section 2 the cycle has five categories: $A = \{13, 14\}$, $B = \{9, 10, 11, 12\}$, $C = \{4, 7, 8\}$, $D = \{3, 5, 6\}$, $E = \{1, 2\}$. Card 13 (A) beats cards 3, 5 and 6 (D), cards 9 and 10 (B) beat card 1 (E), and cards 11 and 12 (B) beat card 2 (E). These are losses to a card three categories up, which conditions (3) and (4) forbid.

Spivey had left room for this. After characterizing cycles with three categories (Corollary 15 of [1]), he wrote that a version for five categories “would be more difficult to prove (and might not even be true)”, partly because “a player is not guaranteed to win two consecutive skirmishes”, and he gave an AD against BC matchup as the example [1, p. 762]. This cycle shows that such a version fails. It has five categories, and its runs of wins have lengths 1 and 2, not the runs of eight suggested by Corollary 15 (runs of two for three categories) and Spivey’s remark on four categories (runs of four). Corollary 6.8 below gives such cycles with every odd number of categories from 5 on.

Here the choice between Spivey’s two labelings matters. Under the numerical labeling of his Section 3 the same cycle has three categories, $\{13, 14\}$, $\{3, 5, 6, 9, 10, 11, 12\}$ and $\{1, 2, 4, 7, 8\}$, and card 4, in the third, beats card 3, in the second. Corollary 15 states that a cycle has three categories exactly when the players alternate pairs of wins, and its proof uses the fact that C cards never win. That is true for the definition of Section 2, under which no card beats a card of a higher category. It is false for the numerical labeling, under which this cycle would contradict Corollary 15. So Corollary 15 is a statement about the first definition.

6.2 An infinite family

In category letters the 96-turn cycle for $n = 14$ is a short word pattern, and the pattern extends to larger decks. For a word w write w^k for k copies of w .

Theorem 6.2. *For every $k \geq 1$, the letter position*

$$\text{player 1: } (\text{ABBC})^k \quad \text{player 2: DE (ABBC)}^k \text{ CDDE}$$

with $n = 8k + 6$ cards ($2k$ A, $4k$ B, $2k + 1$ C, three D, two E) repeats after exactly $4k + 4$ turns, and two equal letters never meet. Hence War with $n = 8k + 6$ cards has a cycle that is not of Theorem-3 type.

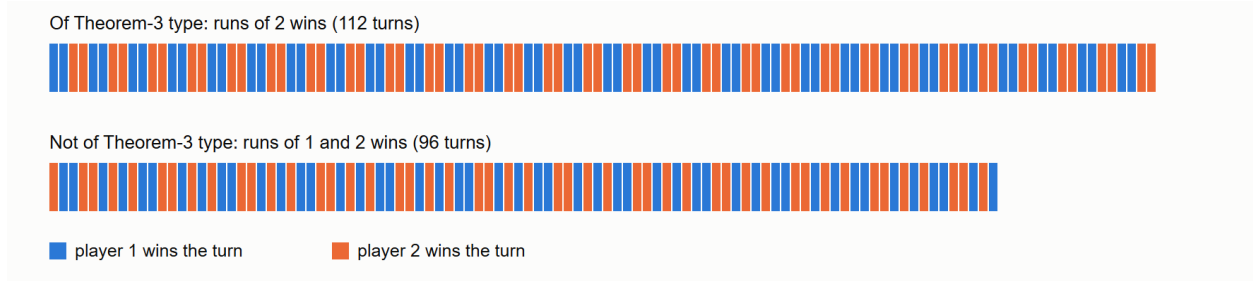


Figure 1: Who wins each turn, once around two 14-card cycles. Top: a cycle of Theorem-3 type (length 112), in which the players alternate runs of two wins. Bottom: a cycle of length 96 that is not of Theorem-3 type.

Proof. For $0 \leq i \leq k-1$ let $S(i)$ be the position

$$\text{player 1: } BC(ABBC)^{k-1-i}ADBE(ABBC)^i \quad \text{player 2: } (ABBC)^{k-i}CDDE(ABBC)^i.$$

We write $X > Y$ for a turn in which letter X beats letter Y , and name the player who wins it.

From the start to $S(0)$, two turns. $A > D$ and $B > E$, both won by player 1, who puts AD and then BE under the pile. This is $S(0)$.

From $S(i)$ to $S(i+1)$, four turns, for $i \leq k-2$. Player 1 starts with $BCAB$ and player 2 with $ABBC$. The turns are $A > B$ and $B > C$ won by player 2, then $A > B$ and $B > C$ won by player 1. Player 1 plays $BCAB$ from the front, player 2 plays one copy of $ABBC$, and each appends $ABBC$. The result is $S(i+1)$.

From $S(k-1)$ back to the start, six turns. The position is

$$\text{player 1: } BCADBE(ABBC)^{k-1} \quad \text{player 2: } ABBCCDDE(ABBC)^{k-1}.$$

The turns are $A > B$ (player 2), $B > C$ (player 2), $A > B$ (player 1), $C > D$ (player 2), $B > C$ (player 1) and $D > E$ (player 2). They leave player 1 with $(ABBC)^k$ and player 2 with $DE(ABBC)^kCDDE$, the starting position.

The letter cycle has length $2 + 4(k-1) + 6 = 4k + 4$. Every turn is between different letters and neither pile empties, so by Spivey's proof of Lemma 1 (Section 2) the numbered game never ends and follows these winners. In the last six turns the winners go 2, 2, 1, 2, 1, 2, so runs of length 1 occur next to runs of length 2, and condition (1) fails. All the other wins belong to runs of length 2. So the four single wins occur once in each $4k + 4$ turns, and the position cannot repeat sooner. $\square$

Figure 2 shows the case $k = 3$. Most of each pile is copies of $ABBC$, and the words $ADBE$ and $CDDE$ form a defect that moves one copy further every four turns. The first member, $k = 1$, is the 14-card cycle above. It accounts for all 720 cycles of 14 cards not of Theorem-3 type: its letter cycle has 8,640 numberings that respect every meeting, each cycle arises from 12 of them, and the 720 cycles obtained are exactly those found by the exhaustive search (program `family_tokens.py`). Every intermediate position $S(i)$ and the length $4k + 4$ of the letter cycle were also checked by machine for $k \leq 40$. When every letter beats every later letter, each card's Spivey category equals its letter. Theorem 6.7 with $r = 0$ proves this for every k , and `doubled_family.py` checks it for $k \leq 16$. The numbered games for $k \leq 12$ ($n \leq 102$) were also played (program `family_construct.py`), and their lengths are listed in Section 6.5.



Figure 2: The member $k = 3$ ($n = 30$) of the family in Theorem 6.2, with each card shown by its category. The outlined defect moves one copy of ABBC further every four turns. After 16 turns the letter position repeats (the numbered cycle has length 192).

6.3 All $n = m \cdot 2^v$ with $m \equiv 3 \pmod{4}$, $m \geq 7$, $v \geq 1$

Spivey's construction doubles a cycle: after each card it inserts a card of the category directly below. On words, with A the strongest letter and $w + d$ meaning every letter of w moved d letters down, this is

$$\mathcal{D}(w) = w_1 (w_1 + 1) w_2 (w_2 + 1) \cdots,$$

the word w interleaved with $w + 1$. The family of Theorem 6.2 already comes from doubling. Its start is $\mathcal{D}$ applied to the letter cycle $(AB)^k$ against $D(AB)^k CD$, which has $4k + 3$ letters and in which the players alternate single wins. We call this letter cycle the *base* of the family. Spivey shows that j doublings of a cycle in which the players alternate give runs of 2^j wins [1, p. 751], and his argument uses the fact that in his cycles every meeting is between adjacent letters: the winner of x against $x + 1$ collects $x, x + 1, x + 1, x + 2$, which is $\mathcal{D}(x(x + 1))$. In the base, A also meets D. The doubled game then collects A, D, B, E instead of $\mathcal{D}(AD) = ABDE$, and these meetings produce the runs of single wins. Doubling the family again gives cycles not of Theorem-3 type for many more n .

Theorem 6.3. *For every $j \geq 0$ and $k \geq 1$, War with $n = 2^j(8k + 6)$ cards has a cycle that is not of Theorem-3 type. These n are exactly the numbers $m \cdot 2^v$ with m odd, $m \equiv 3 \pmod{4}$, $m \geq 7$ and $v \geq 1$.*

Proof. Fix j and k , and let $T = \mathcal{D}^j(ABBC)$, of length $|T| = 4 \cdot 2^j$, with halves $H = \mathcal{D}^j(AB)$ and $\mathcal{D}^j(BC) = H + 1$. Let $Y = \mathcal{D}^j(CDDE)$, $Z = \mathcal{D}^j(DE)$, and let X be H interleaved with $H + 3$. The proof uses two facts throughout: doubling commutes with shifting, $\mathcal{D}^j(w + d) = \mathcal{D}^j(w) + d$, and a word u interleaved with $u + 1$ is $\mathcal{D}(u)$. Start from

$$\text{player 1: } T^k \quad \text{player 2: } Z T^k Y.$$

The play splits into phases of $|T|$ turns. In each phase each player plays $|T|$ cards from the front of their pile. The cards put under the piles during a phase are not reached in it, because each pile holds at least $|T|$ cards at the start of every phase. So each phase is a fixed computation on words.

First phase. The two players play, letter against letter,

$$T = \mathcal{D}^j(A) \mathcal{D}^j(B) \mathcal{D}^j(B) \mathcal{D}^j(C) \quad \text{against} \quad ZH = \mathcal{D}^j(D) \mathcal{D}^j(E) \mathcal{D}^j(A) \mathcal{D}^j(B).$$

For 2^{j+1} turns player 1's letter is three letters stronger, so player 1 wins and appends X . For the next 2^{j+1} turns player 2's letter is one letter stronger, so player 2 wins and appends $\mathcal{D}^{j+1}(A)\mathcal{D}^{j+1}(B) = T$. The position is now $(T^{k-1}X, (H+1)T^{k-1}YT)$.

Middle phases. From $(T^{k-1-i}XT^i, (H+1)T^{k-1-i}YT^{i+1})$ with $i \leq k-2$, player 1 plays T and player 2 plays $(H+1)H$. Each player wins 2^{j+1} consecutive turns by one letter and appends T , which gives the same form with $i+1$.

Last phase. From $(XT^{k-1}, (H+1)YT^k)$ player 1 plays X , and player 2 plays $H+1$ followed by the first half $\mathcal{D}^j(CD)$ of Y . Together these are $\mathcal{D}^j(BCCD) = \mathcal{D}^{j+1}(BC)$, that is, $H+1$ interleaved with $H+2$. Turn by turn the letters are H_t against H_t+1 , which player 1 wins, and H_t+3 against H_t+2 , which player 2 wins, so the players alternate single wins. Player 1 appends $\mathcal{D}(H) = T$, player 2 appends $\mathcal{D}(H+2) = Y$, and what is left of player 2's Y is Z . This is the starting position.

Two equal letters never meet and no pile empties. So when the cards are numbered so that every letter beats every later letter, the game follows the letter game move for move, and its starting position lies on a cycle (Section 2). The winners come in runs of 2^{j+1} in the first and middle phases and in runs of 1 in the last phase, so condition (1) fails. Finally $2^j(8k+6) = 2^{j+1}(4k+3)$. $\square$

For $j=0$ this is Theorem 6.2. We also checked every phase identity by machine: for $j \leq 6$ and $k \leq 8$ with `doubled_phases.py`, and for $j \leq 5$ and $k \leq 6$ with a second, independently written program, `doubled_family.py`. A third program, `doubled_check.py`, checks the resulting letter cycle over the same range: no two equal letters meet, the letter cycle has length $(k+1)|T|$, and the runs of wins have lengths 1 and 2^{j+1} .

Ordinary deals reach these cycles.

Theorem 6.4. *For every $j \geq 0$ and $k \geq 1$, some deal of $n = 2^j(8k+6)$ cards, $n/2$ to each player, enters the cycle of Theorem 6.3 after 2^j turns.*

Proof. Along the cycle player 1 holds between $k|T|$ and $k|T| + 2^{j+1}$ cards, fewer than $n/2 = k|T| + 3 \cdot 2^j$, so no position of the cycle is a deal. After the first 2^{j+1} turns of the first phase, player 1 holds $k|T| + 2^{j+1}$ cards and nothing has been added to player 2's pile yet, so that pile ends with $Y = \mathcal{D}^{j+1}(CD)$: 2^{j+1} pairs, each a card followed by a card it beats. As Spivey does for his own cycles [1, p. 751], undo turns won by player 2. Take the last two cards x, y of player 2's pile, put x back on top of that pile and y on top of player 1's. This is possible 2^j times in a row. Each undo moves one card to player 1, so the result has $n/2$ cards on each side. It is a deal, and playing it forward repeats the 2^j undone turns and reaches the cycle. $\square$

Two independently written programs checked this with numbered cards: `reach_family.py` for $j \leq 5$ and $k \leq 5$ ($n \leq 1472$), and `reach_check.py` for $j \leq 4$ and $k \leq 4$ ($n \leq 608$).

6.4 More families

The sampling of Section 6.5 finds cycles not of Theorem-3 type at $n = 22$ in six classes, grouped by cycle length and category sizes. The cycles of one class follow the letter cycle of Theorem 6.2. Two more families account for three others, and the second of them shows that these cycles can have arbitrarily many categories.

Theorem 6.5. *For all $s \geq 0$ and $t \geq 1$ the letter position*

$$\text{player 1: } (\text{ABBCADBE})^s \text{ ABBC} \quad \text{player 2: } \text{DE} (\text{ABBCCDDE})^t$$

with $n = 8(s + t) + 6$ cards returns to itself after exactly 8 turns, and two equal letters never meet. No cycle that follows it is of Theorem-3 type. For each $n = 8N + 6$ the N positions with $s + t = N$ give N different letter cycles.

Proof. In 8 turns player 1 plays ABBCADBE and player 2 plays DEABBCCD (if $s = 0$, player 1 plays its ABBC and then the AD and BE it wins in the first two turns). The turns are $A > D$, $B > E$ (player 1), $A > B$, $B > C$ (player 2), $A > B$ (player 1), $C > D$ (player 2), $B > C$ (player 1) and $D > E$ (player 2). Player 1 puts ADBEABBC under its pile and player 2 puts ABBCCDDE, which gives back both piles. The winners 1, 1, 2, 2, 1, 2, 1, 2 have runs of lengths 1 and 2, so condition (1) fails. This sequence has no shorter period, so the position does not return sooner. The positions with $s + t = N$ contain $s + N + 1$ cards A, so they are different. $\square$

The case $s = 0$, $t = 1$ is the letter cycle of Theorem 6.2 with $k = 1$. At $n = 22$ the cycles of the classes of lengths 240 and 672 follow the letter cycles for $(s, t) = (0, 2)$ and $(1, 1)$ (Section 6.5).

For the second family we use the shift of Section 6.3: $w + d$ is the word w with every letter moved d letters down. For $r \geq 0$ let

$$U_r = \prod_{i=0}^{r-1} A(D+2i)B(E+2i), \quad V_r = \text{DE} + 2r, \quad W_r = \prod_{i=0}^r (\text{CDDE} + 2i),$$

where the products are concatenations in increasing i . So U_0 is empty, $U_1 = \text{ADBE}$, $V_0 = \text{DE}$, $V_1 = \text{FG}$, $W_0 = \text{CDDE}$ and $W_1 = \text{CDDEEFFG}$.

Theorem 6.6. *For all $r \geq 0$ and $k \geq 1$ the letter position*

$$\text{player 1: } (\text{ABBC})^k U_r \quad \text{player 2: } V_r (\text{ABBC})^k W_r$$

with $n = 8(k + r) + 6$ cards and $2r + 5$ letters returns to itself after exactly $4(k + r + 1)$ turns, and two equal letters never meet. No cycle that follows it is of Theorem-3 type. For $r = 0$ this is Theorem 6.2.

Proof. The play has the three phases of the proof of Theorem 6.2. As there, every card played in a phase is already in place at its start, so no card put under a pile is reached within the same phase. In the first two turns player 1's A and B beat $V_r = (\text{D}+2r)(\text{E}+2r)$, and player 1 puts $A(\text{D}+2r)B(\text{E}+2r)$ under its pile, so that it holds $\text{BC}(\text{ABBC})^{k-1}U_{r+1}$. The next $k - 1$ phases of four turns are those of Theorem 6.2: player 1 plays BCAB, player 2 plays ABBC, and each puts ABBC under its pile. In the last phase, of $4r + 6$ turns, player 1 plays $\text{BC}U_{r+1}$. Player 2 plays

ABBC and then all of W_r except its last two letters, that is, the letter $A + \lfloor t/2 \rfloor$ in turn t . Turns 1 and 2 are $A > B$ and $B > C$ for player 2. Then for $i = 0, \dots, r$ the four turns $4i + 3, \dots, 4i + 6$ are

$$A > B+2i, \quad C+2i > D+2i, \quad B > C+2i, \quad D+2i > E+2i,$$

won by player 1, player 2, player 1 and player 2. Player 1 puts ABBC U_r under its pile and player 2 puts ABBC W_r . The last two letters of W_r , which were not reached, are the new V_r . This is the starting position, after $2 + 4(k-1) + 4r + 6 = 4(k+r+1)$ turns. In the last phase the winners go 2, 2 and then 1, 2 repeated $2(r+1)$ times. With the two wins of player 1 that follow, the runs of wins have lengths 1 and 2, so condition (1) fails. The single wins occur only in the last phase, so the position does not return sooner. $\square$

Theorem 6.7. *Number the cards of the cycle of Theorem 6.6 so that every letter beats every later letter. Then each card's category is its letter, so the cycle has exactly $2r + 5$ categories.*

Proof. An A card never loses, and a card is beaten only by cards of stronger letters. So by induction along the letters no card's category is lower than its letter. For the reverse it is enough, again by induction, that every card of a letter other than A is beaten somewhere in the cycle by a card of the letter directly above.

A B or C in the third or fourth place of a copy of ABBC loses to an A or a B when it is played, with one exception. Player 2's C in the fourth place of its last copy beats a D in turn 4 of the last phase and goes to the first place of the new W_r , where one pass later it loses to a B. The only other C card outside the copies of ABBC is the first letter of W_r , which loses to a B in turn 5 of the last phase.

A B in the second place of a copy wins and goes to the third place of a copy, except for player 1's first copy in the first two turns, whose B goes to the third place of the new block of U_{r+1} . Number the blocks $A(D+2i)B(E+2i)$ of U_{r+1} by $i = 0, \dots, r$. A B in the third place of block i wins in the last phase and goes to the third place of block $i-1$ of the appended U_r , or, if $i = 0$, of the appended ABBC. So it moves one block per pass until it reaches a copy of ABBC, where it next loses to an A.

That leaves the letters from D on. In player 1's pile they are the second and fourth letters of the blocks of U_{r+1} , which in the last phase lose to the letter directly above. Player 2's cards of these letters are in V_r , and go to U_{r+1} in the first two turns, or they are in W_r . There a played card either loses and goes to U_r , or wins and moves to a place of the new W_r that loses at its next play or becomes part of V_r . So within two passes every card of letter D or later is in player 1's U part, where at its next play it loses to the letter directly above. $\square$

Corollary 6.8. *For every $N \geq 1$ War with $n = 8N + 6$ cards has a cycle not of Theorem-3 type with exactly $2N + 3$ categories (Theorems 6.6 and 6.7 with $r = N - 1$, $k = 1$). In particular such cycles have unboundedly many categories, while a cycle of Theorem-3 type whose runs of wins have length 2^v has $v + 2$.*

With the cards numbered so that every letter beats every later letter, ordinary deals reach these cycles too. After the first two turns of the cycle of Theorem 6.6 player 1 holds $n/2 - 1$ cards, and player 2's pile ends with $(D+2r)(E+2r)$. Undoing one win of player 2, as in the proof of Theorem 6.4, gives a deal with $n/2$ cards each that enters the cycle after one turn (checked for $r, k \leq 4$ by `more_families.py`).

The count depends on the definition of categories. Under the numerical labeling of Spivey's Section 3, each of the 19 letter cycles of Theorems 6.5 and 6.6 with $n \leq 38$, numbered so that every letter beats every later letter, has three categories, as the 14-card cycle does (Section 6.1; program `more_families.py`).

At $n = 22$ the cycles of the class of length 360 with categories of sizes 3, 5, 3, 3, 3, 3, 2 follow the letter cycle with $r = k = 1$. The two remaining classes at $n = 22$, of lengths 1,440 and 360 (sizes 2, 3, 3, 5, 4, 3, 2), are reached from doublings of 11-card letter cycles. $\mathcal{D}$ applied to the letter cycle ADCDAD against BCDAB leads into the first after 11 turns, and applied to ADCF against BCDEFAD, which uses six letters, into the second after 13 turns (program `more_families.py`). We have not found families through them. All the families so far, and the doublings of Theorem 6.3, lie at values of n that Conjecture 6.14 allows. So they are consistent with it, but they do not test it.

6.5 Lengths of cycles not of Theorem-3 type

Theorem 5.5 covers every cycle of Theorem-3 type, and hence every cycle for odd n . The cycles constructed above are not of that type, but every cycle not of that type that we have found obeys the same rule.

Conjecture 6.9. *For every n , every cycle of War with n cards has length divisible by $n + 2^v$, where 2^v is the largest power of 2 dividing n (OEIS A006519).*

By Theorem 5.5 the conjecture is open only for cycles that are not of Theorem-3 type. It holds for the cycles that follow the letter cycles of Theorems 6.3, 6.5 and 6.6 (Theorems 6.10 and 6.13 and Corollary 6.12). The proof of Theorem 5.5 used the block structure, which these cycles lack, since their winners come in runs of different lengths. Instead we follow the weakest cards. By Section 2 the length of a cycle is a multiple of the period of each of its cards.

Theorem 6.10. *In every cycle that follows the letter cycle of Theorem 6.3 (for $j = 0$, of Theorem 6.2), the two cards of the weakest letter have period exactly $n + 2^v$, and the length of the cycle is a multiple of $n + 2^v$.*

Proof. Give the cards letters as in the definition of following (Section 2). With each card replaced by its letter the cycle is played exactly as the letter game. Exchanging the players changes no period, so we may assume that the cycle passes through the starting position. Let $R = (k+1)|T| = 2^{j+2}(k+1)$. By the proof of Theorem 6.3 the letter position is back at the start after R turns, and not sooner: in every R turns the runs of single wins form one stretch, the $|T|$ turns of the last phase, and a shift by a proper divisor of R would move that stretch off itself. So a pass takes R turns. Since $n = 2^{j+1}(4k+3)$ with $4k+3$ odd, $2^v = 2^{j+1}$ and $2R = 2^{j+1}(4k+4) = n + 2^v$.

The word $\mathcal{D}^j(x)$ consists of the letters x to $x+j$ and contains $x+j$ once, so the weakest letter $E+j$ occurs exactly twice: as the last card of Z , call it ε , and as the last card of Y , call it ε' . Follow these two cards through one pass, with the phases in the proof of Theorem 6.3. In the first phase, ε is the last card player 2 plays during player 1's run of 2^{j+1} wins. It loses and becomes the last card of X . Player 1 does not reach X again until the last phase, whose last turn is $H_t + 3$ against $H_t + 2$ with ε as the card $H_t + 3$. It loses there and ends as the last card of player 2's pile, where ε' started. The card ε' is never played. Y is not reached in the first and middle phases, and in the last phase player 2 plays only its first half, so ε' ends as the last card of the new Z , where ε started. So after R turns the letters are back at the start, with the two weakest cards exchanged.

At every multiple of R the letter position is the start, and the cycle follows the letter game, so where a card goes in the next R turns depends only on its place. Hence each of the two weakest cards is back in its place after $2R$ turns, and its period divides $2R$. In the turns $0, \dots, 2R - 1$ it follows the path of ε in one half and the path of ε' in the other, so it is played at least once in one half and never in the other. Whether it is played at turn t depends only on its place at turn t . So every p consecutive turns, with p its period, contain the same number of its plays, and this number is positive. A divisor of $2R$ smaller than $2R$ is at most R , so a window of p turns would then fit inside the half with no plays. Therefore $p = 2R = n + 2^v$. $\square$

Card 1 always loses, so it is a card that never wins. In a numbering that respects every meeting of the letter cycle, any card that never wins can be card 1. It need not have the weakest letter: for $n = 22$ two C cards also never win. The next result covers all such cards.

Theorem 6.11. *In a cycle as in Theorem 6.10, the cards that never win are the two cards of the weakest letter (the last cards of Z and of Y at the start) and the last card of each of the first $k - 1$ copies of T in each player's pile at the start. Each of them has period exactly $n + 2^v$. In particular card 1 has period exactly $n + 2^v$ in every cycle that follows the letter cycle of Theorem 6.3 (for $j = 0$, of Theorem 6.2).*

Proof. At the start player 1 holds copies $T_1, \dots, T_k$ of T , and player 2 holds Z , copies $T'_1, \dots, T'_k$ and Y . Places run from 1 to $|T|$ in a copy of T and in Y , and from 1 to $|T|/2$ in Z . Let σ send a card's place at the start of a pass to its place at the end of the pass. In each phase each player appends at most one word of $|T|$ cards, and in it the winner of each turn comes first. So the loser of the winner's t -th win in a phase lands in place $2t$ of the word appended in that phase. Following the phases in the proof of Theorem 6.3 card by card: a card in the first half of a copy of T wins when it is played; a card in place $|T|/2 + t$ of T_i (any i), or of T'_i with $i \leq k - 1$, loses and goes to place $2t$ of the other player's new i -th copy; a card in place $|T|/2 + t$ of T'_k is played at turn t of the last phase, and wins if t is even, and otherwise goes to place $t + 1 \leq |T|/2$ of player 1's new k -th copy; a card in place u of Z loses in the first phase and again at turn $2u$ of the last phase, and goes to place $2u$ of the new Y ; a card in place $y \leq |T|/2$ of Y is played at turn $|T|/2 + y$ of the last phase, and wins if y is even, and otherwise goes to place $|T|/2 + y + 1$ of player 1's new k -th copy; and a card in place $y > |T|/2$ of Y is not played and goes to place $y - |T|/2$ of the new Z .

Call a place *winning* if the card in it at the start of a pass wins at least once during the pass. A card never wins exactly when no place in its orbit under σ is winning. In copies $i \leq k - 1$ a card in place $|T|/2 + t$ with $t < |T|/2$ moves to $2t < |T|/2 + t$, so it reaches a first half and wins, while the last cards of T_i and T'_i exchange places every pass, each played once. In copy k , a card of T'_k wins or moves to a first half. A card of T_k in place $|T|/2 + t$ goes to place $2t$ of T'_k , which is in the first half or is $|T|/2 + (2t - |T|/2)$ with $2t - |T|/2$ even, so it wins too. In Z , a card in place $u < |T|/2$ goes to $2u$ in Y and, if $2u > |T|/2$, back to $2u - |T|/2 < u$ in Z , so it reaches a winning place. The card ε , in place $u = |T|/2$, goes to the last place of Y , and ε' goes to place $|T|/2$ of Z . A card of Y at odd $y \leq |T|/2$ goes into the second half of T_k and wins later, and one at $|T|/2 < y < |T|$ goes to Z below $|T|/2$. So the cards that never win are the ones listed. For each of them σ exchanges its place with one other place, and the card is played exactly twice in $2R$ turns.

Let p be the period of such a card. It divides $2R$, and every p consecutive turns contain the same number of its plays, so $2R/p$ divides 2 and p is R or $2R$. It is not R , since σ moves the card. So $p = 2R = n + 2^v$. Card 1 never wins, so it is one of these cards. $\square$

For every member with $n \leq 255$ the program `family_tokens.py` computes the cards that never

win directly from the letter game. They are exactly the ones listed, and each has period exactly $n + 2^v$.

Corollary 6.12. *In every cycle that follows the letter cycle of Theorem 6.6, the two cards of the weakest letter $E + 2r$ have period exactly $n + 2$, and the length of the cycle is a multiple of $n + 2 = n + 2^v$. When every letter beats every later letter, these are cards 1 and 2.*

Proof. The argument of Theorem 6.10 applies with $R = 4(k + r + 1)$, the length of the letter cycle, so that $2R = 8(k + r + 1) = n + 2$. Also $2^v = 2$, since $n \equiv 2 \pmod{4}$. The weakest letter occurs twice, as the last cards of V_r and of W_r . By the proof of Theorem 6.6, the first loses in the first two turns and becomes the last card of the new block of U_{r+1} . Player 1 does not reach it again until the last turn of the last phase, where it loses and ends as player 2's last card. The second is never played and ends as the last card of the new V_r . So after R turns the two cards have exchanged places, and each is played in one pass of every two and not in the other. The second paragraph of the proof of Theorem 6.10 then gives period exactly $2R$. When every letter beats every later letter, the weakest letter holds the numbers 1 and 2. $\square$

For the family of Theorem 6.5 the weakest letter behaves differently.

Theorem 6.13. *In every cycle that follows the letter cycle of Theorem 6.5, the cards that never win are exactly the $s + t + 1$ cards of letter E, and each of them has period exactly $n + 2^v = 8(s + t + 1)$. So card 1 has period exactly $n + 2^v$, and the length of the cycle is a multiple of $n + 2^v$.*

Proof. Write player 1's pile at the start of a pass as $G_1 \cdots G_s T$ with $G_i = \text{ABBCADBE}$ and $T = \text{ABBC}$, and player 2's as $\text{DE } G'_1 \cdots G'_t$ with $G'_i = \text{ABBCCDDE}$. Places in a copy run from 1 to 8, and the turns of a pass are numbered 1 to 8. Let σ be the place map of a pass, as in the proof of Theorem 6.11. By the proof of Theorem 6.5 a pass plays G_1 (or, if $s = 0$, T and the four cards won in its first two turns), DE and the first six cards of G'_1 . Unplayed cards move 8 places forward, and the winner of each turn goes under its pile first. For $s \geq 1$, after the pass the old T followed by the appended ADBE is the new G_s , the appended ABBC is the new T , the last two cards of G'_1 are the new DE , and the appended ABBCCDDE is the new G'_t . Hence the E in DE loses in turn 2 and becomes the last card of the new G_s . The E of G_i moves to G_{i-1} , and that of G_1 loses in turn 8 and becomes the last card of the new G'_t . The E of G'_i moves to G'_{i-1} , and that of G'_1 becomes the E of the new DE . If $s = 0$, the E of DE loses in turns 2 and 8 and becomes the last card of the new G'_t , and the other E cards move as before. So the $s + t + 1$ places of the E cards form one orbit of σ .

Every other card reaches a winning place. The cards in places 1, 2, 5, 7 of G_1 and in places 1, 2, 4, 6 of G'_1 win. A losing card of another letter goes from G_1 to place 2, 4 or 6 of the new G'_t , from G'_1 to place 2 or 4 of the new T , or, for the D of DE , to place 6 of the new G_s . From each of these places it moves forward to a winning place of G'_1 or G_1 . (A C from place 4 of T reaches place 4 of G_1 , loses again and then wins from place 4 of G'_1 . The D from place 6 of G_s reaches place 6 of G_1 , loses, and wins from place 6 of G'_1 .) If $s = 0$ the same holds. Here the A and B of T win, go to the appended ADBE and win again. The B and C of T go to places 2 and 4 of the new G'_t , and the D of DE loses in turn 6, goes to place 6 of the new G'_t and wins from G'_1 .

Follow an E card from DE at the start of a pass, and count turns from there. It is played in turns 2 and $8s + 8$: the second time in turn 8 of pass $s + 1$, when it has reached G_1 , or, for $s = 0$, in the first pass, via the appended ADBE . It is back in DE after $8(N + 1)$ turns, $N = s + t$, without being played in between. Its period p divides $8(N + 1)$, and every p consecutive turns contain the same number of its plays, so p is $8(N + 1)$ or $4(N + 1)$. In the second case the two plays would be

$4(N + 1)$ turns apart, so $8s + 6 = 4N + 4$, which is impossible modulo 4. So $p = 8(N + 1) = n + 2$, and $2^v = 2$ since $n \equiv 2 \pmod{4}$. Card 1 never wins, so it is one of these cards. $\square$

By Section 2 all cycles that follow one letter cycle have the same length. For the letter cycles of Theorem 6.2 with $k \leq 12$ ($n = 14, 22, \dots, 102$) the lengths are 96, 288, 192, 960, 1,440, 672, 2,688, 3,456, 1,440, 5,280, 6,336 and 2,496. Those of Theorem 6.3 include 3,360 ($n = 28$), 720 ($n = 44$), 349,440 ($n = 56$), 37,440 ($n = 60$), 822,528 ($n = 88$), $219,098,880 = 128 \cdot 1,711,710$ ($n = 112$) and 4,053,504 ($n = 120$). We simulated all 57 members of Theorem 6.3 with $n \leq 255$ (program `cycle_length.c`). The longest cycle is 5,855,351,040 turns ($n = 240$), and every length is a multiple of $n + 2^v$, as Theorem 6.10 requires.

For $n \leq 16$ nothing lies outside these families. By the exhaustive search the only cycles not of Theorem-3 type have 14 cards, and all 720 of them follow the letter cycle of Theorem 6.2 with $k = 1$ (Section 6.2) and have length $96 = 6 \cdot 16$. With Theorem 5.5 this gives a computer-assisted proof of the conjecture, and of the exact period of card 1, for every cycle with at most 16 cards.

Beyond 16 cards, for cycles that follow none of the letter cycles of Theorems 6.3, 6.5 and 6.6, the conjecture rests on computation. We sampled 10^7 random positions for each of 14 values of n from 18 to 52 (program `sampler.c`, with seed n). None of them reaches a cycle for $n = 24, 40, 48, 52$. For $n = 24$ and 48, of the form $3 \cdot 2^k$, this says little about Question 2, since at $n = 40$ and 52, where cycles exist, the samples found none either. For the other ten values of n (18, 20, 22, 26, 28, 34, 36, 42, 44, 50), a total of 291,306 positions reach a cycle (many of them the same cycle), and every cycle reached has length a multiple of $n + 2^v$.

Only at $n = 22$ do some of them reach cycles not of Theorem-3 type: 318 positions. Grouped by the length of the cycle and the sizes of its categories (A first), 194 reach cycles of length 288 with sizes 4, 8, 5, 3, 2, and every one of these cycles follows the letter cycle of Theorem 6.2 with $k = 2$. The other 124 reach cycles that do not, since their category sizes differ. With each card replaced by its category, all cycles of one class have the same letter cycle, and Section 6.4 accounts for them. Of these, the 50 positions in the class of length 240 (sizes 3, 6, 5, 5, 3) and the 48 in the class of length 672 (4, 7, 4, 4, 3) reach cycles that follow the letter cycles of Theorem 6.5 with $(s, t) = (0, 2)$ and $(1, 1)$. The 5 in the class of length 360 with sizes 3, 5, 3, 3, 3, 2 follow that of Theorem 6.6 with $r = k = 1$. The 15 in the class of length 1,440 (3, 5, 4, 6, 4) and the 6 in the class of length 360 (2, 3, 3, 5, 4, 3, 2) reach the letter cycles entered from doubled 11-card cycles (program `more_families.py`, on all 318 positions). So the theorems above cover 297 of the 318 positions, and only the last two classes, 21 positions, rest on the computation alone. In the cycle reached by each of the 318 positions, card 1 has period exactly $24 = n + 2^v$.

6.6 Which n have cycles not of Theorem-3 type

Conjecture 6.14. *War with n cards has a cycle that is not of Theorem-3 type only if $n = m \cdot 2^v$ with $m \equiv 3 \pmod{4}$, $m \geq 7$ and $v \geq 1$.*

Evidence. For odd n the conjecture is Spivey's Theorem 6. When $n = 2^k$ or $n = 3 \cdot 2^k$ and $n \leq 16$ there are no cycles at all (Theorem 4.1). With $m \equiv 1 \pmod{4}$ and $m \geq 5$, the exhaustive search at $n = 10$ finds only cycles of Theorem-3 type. Among the random positions of Section 6.5, those that reach a cycle (203,657 for $n = 18$, 9,337 for $n = 20$, 23,126 for $n = 26$, 2,654 for $n = 34$, 8 for $n = 36$, 398 for $n = 42$ and 40 for $n = 50$) all reach cycles of Theorem-3 type, while at $n = 22$ exactly 318 of the 50,575 reach cycles not of Theorem-3 type.

An exhaustive search (`base_search.c`) covers all letter cycles of an odd number $m = 5, 7, \dots, 15$ of cards whose letters lie between A and D, with every word and every split. It contains the bases of Theorem 6.2 for $k \leq 3$. In this search, doubling yields cycles not of Theorem-3 type for $m = 7, 11, 15$ and for none of $m = 5, 9, 13$. Doubling the same letter cycles two or three times gives the same split, so within the search the pattern also holds for $v = 2$ and 3. A second, independently written program (`base_enum.c`) reproduces every count of this search. With five or six different letters, again with every word and every split, nothing changes for $m = 5$ and 9: no doubled base gives a cycle not of Theorem-3 type. At $n = 22$ the doublings of all 11-card bases over at most six letters give exactly the six classes found by sampling, and no other, and at $n = 14$ they give only the class of Section 6.1 (`classify_doubled.py`).

Beyond this search the evidence for $v \geq 2$ is thin. At $n = 20$ and 36 every sampled position that reaches a cycle reaches one of Theorem-3 type. But sampling can miss the other kind even where it exists: none of the 1,510 positions at $n = 28$ that reach a cycle reaches one of that kind, although Theorem 6.3 gives such cycles there. And for larger n random positions seldom reach a cycle.

7 Question 3 and the shape of the data

Question 3 asks for the probability that a deal never ends. Table 1 gives it exactly, as $a(n)/n!$, for $n \leq 17$. The odd and even decks behave differently (Figure 3). For odd n the share rises steadily: 25.0%, 45.7%, 60.3%, 70.7%, 78.4%, 83.8% and 87.9% for $n = 5, 7, \dots, 17$. For even $n \leq 16$ the share is zero exactly when n is of the form 2^k or $3 \cdot 2^k$, and small otherwise: 10.9% at $n = 10$ and 4.35% at $n = 14$. Seven values say nothing about a limit (Section 9).

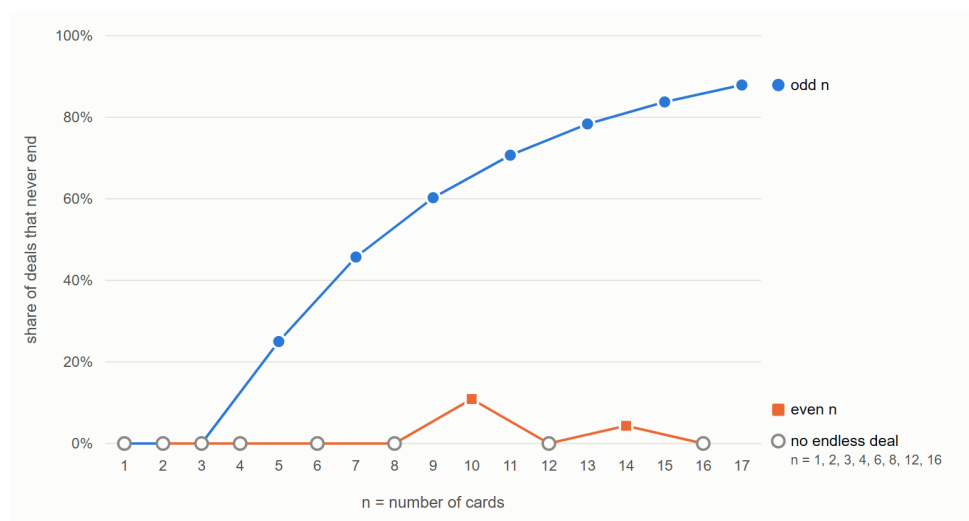


Figure 3: Share of the $n!$ deals whose game never ends. Hollow markers are the zeros at $n = 2^k$ and $3 \cdot 2^k$.

8 Related sequences

The same computations give several related sequences. One is the last column of Table 1, the longest game that ends. We list the others with the programs that produced them.

Distinct cycles. The number of different cycles of n cards, counting a cycle and its exchange as one, is 2, 52, 336, 48, 55,592, 319,680, 26,640 and 48,304,896 for $n = 5, 7, 9, 10, 11, 13, 14, 15$, and 0 for $n \leq 4$ and $n = 6, 8, 12, 16$. For $5 \leq n \leq 14$ two searches agree on these. One is a scan of the canonical positions of Lemma 3.1 by `war_gpu.py`, in which each cycle is weighted by the number of canonical positions it contains. The other is `distinct_cycles.c`, which counts each cycle once, at its smallest canonical position, and records its length. All 48 cycles of 10 cards have length 60, and those of 14 cards have length 112 (25,920 of them) or 96 (720). For odd $n \leq 15$ a third program, `odd_cycles.c`, enumerates only the positions allowed by Spivey’s Theorem 6, plays each one for the length given by Theorem 5.1, and agrees with Theorem 5.2. For $n = 15$, where no scan of the canonical positions was run, this program and the theorem give the two counts. The program’s completeness rests on Spivey’s Theorem 6 and Theorem 5.1. The values for $n = 5, 7, 9$ are the “cycles différents” of Delahaye and Mathieu [2], and their 48 cycles for $n = 10$, found from deals, are all the cycles of 10 cards. Of the 26,640 cycles of 14 cards, 720 are not of Theorem-3 type (Section 6.1). A backward search from each cycle by `reach_cycles.c` (a position has at most two predecessors) shows that for every $n \leq 14$ each of these cycles, or its exchange, is entered by some deal. Counted over deals, 61,184,620 deals of 14 cards fall into the 720 cycles not of Theorem-3 type (Section 6.1). For odd n , Theorem 5.2 gives the sequence up to $n = 41$ (Table 2 and the repository). For even $n \geq 18$ no value is known.

The other rule for collecting cards. If the losing card goes under the winner’s pile first and the winning card second, the number of deals that never end is 0, 2, 0, 64, 72, 3,363, 0, 273,156, 322,940, 32,668,555, 23,880,636, 5,374,605,768, 4,544,440,488, 1,173,142,165,027 for $n = 2, \dots, 15$. These were computed by `war_gpu.py`, `war_gpu_floyd.py` and `loser_first.c` for $n \leq 13$, by these and `war.cpp` for $n = 14$, and by the two GPU programs for $n = 15$. For $2 \leq n \leq 15$ the zeros fall exactly at $n = 2, 4, 8$, the powers of 2 in that range. With this rule both GPU programs also scanned every canonical position for $2 \leq n \leq 12$ and for $n = 16$. Lemma 3.1 does not depend on the order in which the cards are collected, so the scans cover every cycle: no position of 2, 4, 8 or 16 cards cycles, and for every other n from 2 to 16 some deal does. Spivey notes that for this rule a version of his Theorem 3 gives cycles for every n that is not a power of 2 [1, p. 762]. Whether a power of 2 can have cycles is the analogue of Question 2, and up to 16 none can. The counts for $1 \leq n \leq 16$ are A400868 in the OEIS [8].

9 Open problems

1. **Question 2.** Do cycles exist for $n = 2^k$ or $3 \cdot 2^k$? None exist for $n \leq 16$ (Theorem 4.1). No cycle of these n can be of Theorem-3 type (Section 5.3), so Conjecture 6.14 would answer Question 2 in the negative.
2. **The converse of Theorem 6.3** (Conjecture 6.14): cycles not of Theorem-3 type exist only for $n = m \cdot 2^v$ with $m \equiv 3 \pmod{4}$, $m \geq 7$, $v \geq 1$. A proof probably needs an understanding of cycles with five or more categories. Spivey proves his characterization for three categories and outlines it for four [1, p. 762]. The 14-card cycle shows that the five-category analogue fails (Section 6.1), and Corollary 6.8 gives cycles not of Theorem-3 type with every odd number of categories from 5 on.
3. **Cycles not of Theorem-3 type.** How many are there? For $n \leq 16$ there are 720, all at $n = 14$. At $n = 22$ random sampling finds them in six classes. Four follow letter cycles of the

families of Theorems 6.2, 6.5 and 6.6, and the other two are entered from doublings of 11-card letter cycles (Section 6.4). Is there a formula for the number of cycles that follow the letter cycle of Theorem 6.3, like Theorem 5.4 for cycles of Theorem-3 type? At $n = 14$ they are counted by the numberings of the family's start (Section 6.2). And how many cycles lie outside the families?

4. **The lowest card** (Conjecture 6.9 and a stronger form). In every cycle, does card 1 return to the same place after exactly $n + 2^v$ turns? That would imply Conjecture 6.9. It holds for every cycle of Theorem-3 type (Theorem 5.5), and for every cycle that follows the letter cycle of Theorem 6.3 or of Theorem 6.5 (Theorems 6.11 and 6.13). The same is true for the cycles that follow a letter cycle of Theorem 6.6 when numbered so that every letter beats every later letter (Corollary 6.12), and, computer-assisted, for every cycle with at most 16 cards. For all other cycles it is open.
5. **Reachability.** Is every cycle entered by some deal? For $n \leq 14$ it is, counting a cycle and its exchange as one (Section 8), and so are the cycles constructed in Theorem 6.3 (Theorem 6.4). Spivey notes that with more than two players, cycles built as in the proof of his Theorem 3 are not reachable from a deal unless the number of players is a power of 2 [1, p. 762].
6. **Question 3 in the limit.** Does $a(n)/n!$ tend to 1 along odd n ? The cycles themselves are rare. At $n = 15$ they contain 369,699,840 of the $14! \cdot 14 \approx 1.2 \times 10^{12}$ canonical positions, about 3×10^{-4} of them, yet 83.8% of deals fall into one. Theorem 5.2 counts the cycles exactly. What is open is the size of their basins.
7. **More terms.** $a(18)$ needs $18! \approx 6.4 \times 10^{15}$ games, 18 times as many as $a(17)$, and each of the two runs for $a(17)$ took about a week on one GPU (Section 3).

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